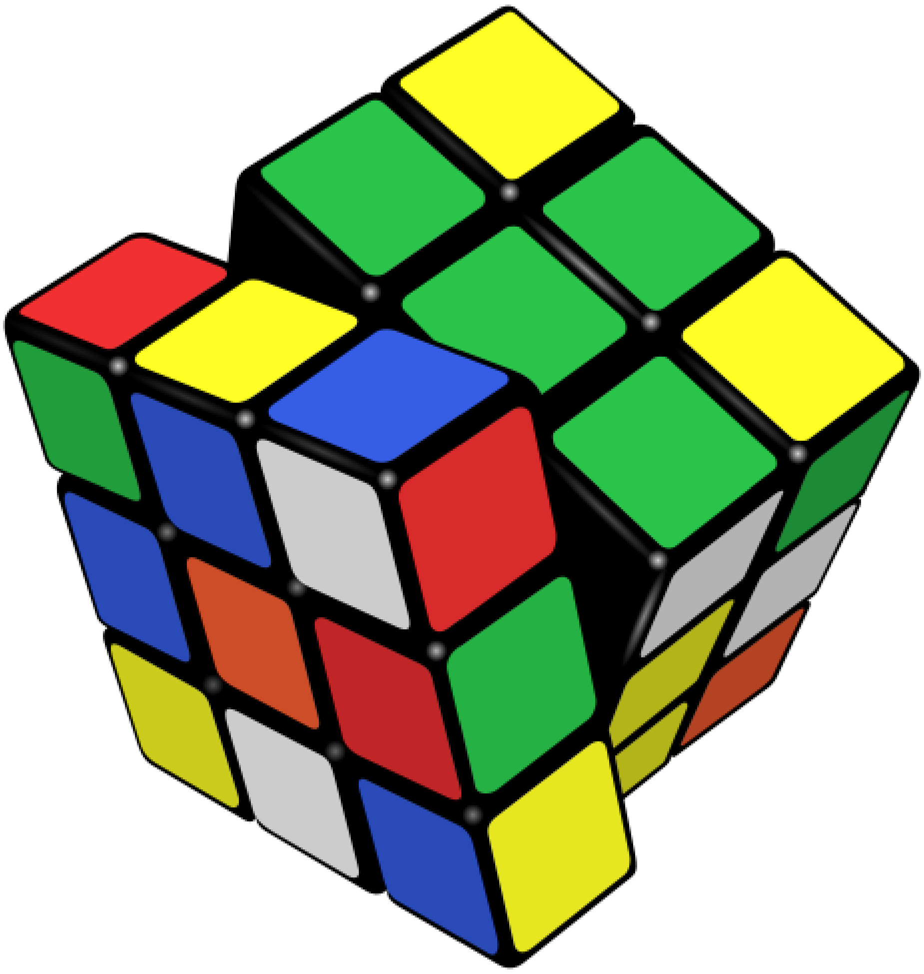
Year 8

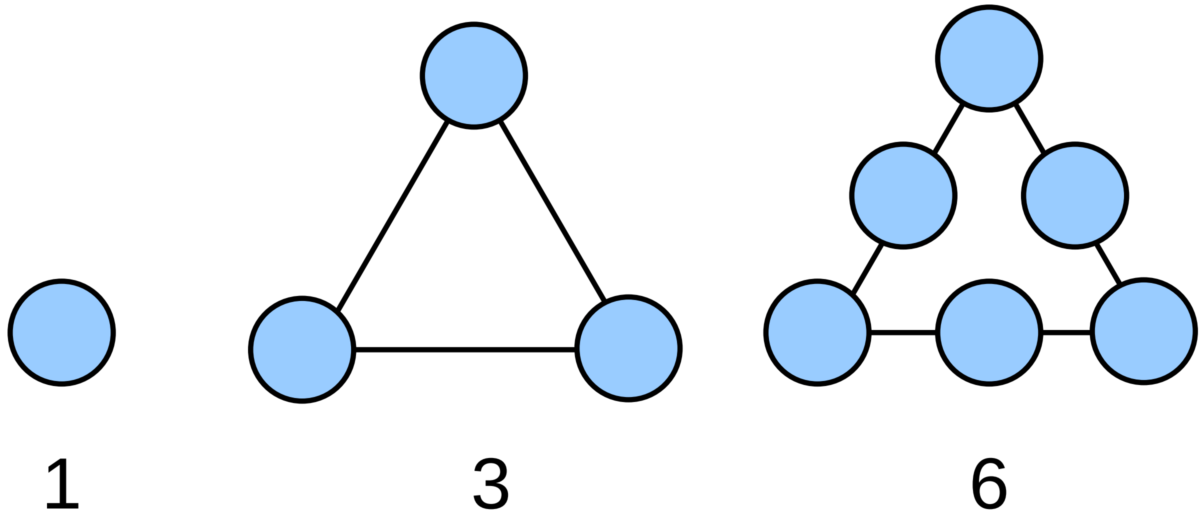


[NUMBER THEORY: FACTORS AND PRIME NUMBERS]

Polygonal Numbers

Triangular Numbers

The first three triangular numbers are shown below.



The pattern continues in much the same way. You will notice that we add two, then three and four and five and so on ad infinitum.

The triangular numbers above could be recorded as follows:

$$T_1=1,\ T_2=3,\ T_3=6,\dots$$

1. Using the notation shown above, work out the triangular numbers to T_{10}.

$$T_n = 1 + 2 + 3 + 4 + \dots + (n-1) + n$$

$$T_n = n + (n-1) + (n-2) + (n-3) + \dots + 2 + 1$$

Above, we have written an equation to calculate the triangular number. We have also written the equation backwards. If we add these together, we get:

$$2T_n = (n+1) + (n+1) + (n + 1) + \dots + (n+1)$$

This gives us n lots of (n+1).

$$\rightarrow 2T_n = n(n+1)$$

$$\rightarrow T_n = \tfrac{1}{2}n(n+1)$$

2. **Calculate the values of the following triangular numbers.**

i) T_{14} ii) T_{23} iii) T_{45} iv) $T_{32} - T_{20}$ v) $T_{25} - T_{24}$ vi) $T_{36} - 2T_{18}$ vii) $T_{50} - 2T_{25}$

viii) Play about with the triangular numbers. Can you find any interesting patterns?

$$T_1 = 1^2,\ T_1 + T_2 = 2^2,\ T_2 + T_3 = 3^2$$

$$T_{(n-1)} + T_n = n^2$$

3. **Are the statements above correct. Check to see if this pattern holds for additions up to and including n=12.**

Hard Extension

H1: The formula, $\mathbf{T_{(n-1)} + T_n = n^2}$, can be proven using algebra. Using the formula for a triangular number twice, see if you can prove it.

4. **The conjecture is suggested that if n is a triangular number, then 9n + 1 is also a triangular number. Check whether this hypothesis holds for values up to n=12.**

Other Types of Polygonal Number

	Number of Sides	Term 1	Term 2	Term 3	Term 4	Term 5	Term 6	Term 7	Term 8
Triangular	3	1	3	6	10	15	21	28	36
Square	4	1	4	9	16	25	36	49	64
Pentagonal	5	1	5	12	22	35	51	70	92
Hexagonal	6	1	6	15	28	45	66	91	120
Heptagonal	7	1	7	18	34	55	81	112	148
Octagonal	8	1	8	21	40	65	96	133	176

5. **Look at the numbers in the table above. Explore the differences between each term on each row and see if there is a pattern. Can you predict the ninth and tenth term for each of the types of polygonal number?**

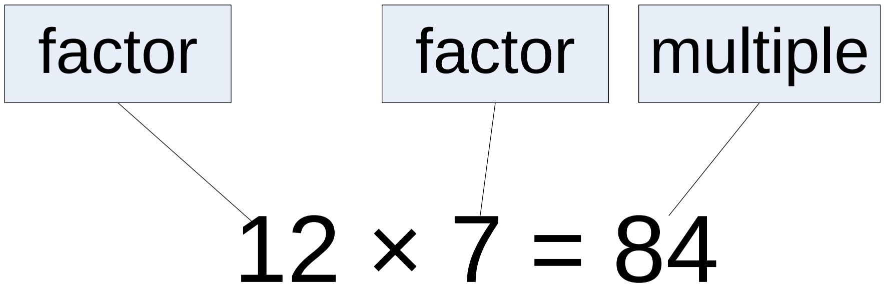
Hard Extension

H2: Can you work out the formulae (in terms of n, where n is the term number) for each row of polygonal numbers.

H3: The triangular numbers 15 and 21 have the property that their sum is 36 and their difference is 6. Both 6 and 36 are also triangular numbers. Find another pair of triangular numbers that have this property.

Factors and Multiples

A factor is part of the question in a multiplication problem. A multiple is the answer.



Numbers have either lots of few factors. Factors are whole numbers or integers.

6. **Calculate all the factors of the numbers below:**

i) 14 ii) 24 iii) 36 iv) 27 v) 31 vi) 56 vii) 1 viii) 96 ix) 150 x) 67

7. **For all the numbers between 30 and 50, calculate all the factors and then draw a chart recording the number of factors that each number has.**

If we have two integers such that a is divisible by b, then we can say that b is a factor of a or b is a divisor of a. In shorthand, we write b|a which means b divides a or b is a factor of a.

eg 2|4 means 2 divides 4 or 2 is a factor of 4.

7|28 means that 7 is a factor of 28.

8. **Find the multiples of the following numbers:**

i) 7, 6, 3 ii) 9, 4, 3 iii) 11, 9, 8 iv) 13, 4, 2 v) 17, 16, 15

9. **Look at each of the answers for question 8. List all of the factors for each multiple.**

If we consider the numbers 18 and 24, the factors for the first of these numbers are 1, 2, 3, 6, 9, 18.

The factors for the latter of these numbers are 1, 2, 3, 4, 6, 8, 12, 24. You will notice that some of these numbers are highlighted. These numbers are common to both the set of numbers that are factors of 18 and the set of numbers that are factors of 24. They are called common factors.

Highest Common Factor

The highest common factor of two integers (say a and b), not both of which are zero, is the positive integer n that satisfies the following:

a) $n|a$ and $n|b$;

b) if $d|a$ and $d|b$, then $d \le n$.

We denote the highest common factor as hcf(a,b).

10. Find the solution to the following:

i) hcf(9,24) ii) hcf(36, 52) iii) hcf(18, 30) iv) hcf(2412, 4929)

The following two facts are worth noting:

$\text{hcf}(0, b) = |b|$

$\text{hcf}(a,b) = \text{hcf}(-a,b) = \text{hcf}(a, -b) = \text{hcf}(-a,-b)$.

11. Find the solution to the following:

i) hcf(-9,24) ii) hcf(36, -52) iii) hcf(27, -42) iv) hcf(8238, -14)

Lowest Common Multiples

As well as having common factors, we also have common multiples.

Since $6|60$ and $10|60$, we can say that 60 is a common multiple of 6 and 10. However, there are infinitely more common multiples of 6 and 10. The positive ones include 30, 60, 90, 120 etc. Of these, 30 is the lowest and so is called the Lowest Common Multiple.

12. Find the lowest common multiple of the following:

i) lcm(8, 20) ii) lcm(6, 14) iii) lcm(3,13) iv) lcm(27,45) v) lcm(39,40)

For any pair of positive integers, a and b:

$$\text{lcm}(a,b) \times \text{hcf}(a,b) = |ab|$$

13. Check the above equation on the following sets of numbers

i) {7, 24} ii) {9, 15} iii) {6,7} iv) {14, 22} v) {12, 21}

The Division Algorithm

If we divide 259 by 74, we get:

$$259 = 3 \times 74 + 37$$

where 3 is the quotient and 37 is the remainder.

14. Divide the following sets of numbers and write the answer in the form of the division algorithm.

i) {3, 542} ii) {8, 3827} iii) {39, 3837} iv) {28, 93822} v) {48, 281921}

An extension of the division algorithm called ***Euclid's Algorithm*** can be used to calculate the highest common factor of two numbers.

eg Find hcf(3108,5291)

$$5291 = 1 \times 3108 + 2183$$

$$3108 = 1 \times 2183 + 925$$

$$2183 = 2 \times 925 + 333$$

$$925 = 2 \times 333 + 259$$

$$333 = 1 \times 259 + 74$$

$$259 = 3 \times 74 + 37$$

$$74 = 2 \times 37 + 0$$

When the remainder =0, we stop the process. The hcf(3108,5291)=37 which is the last non-zero remainder.

15. Use Euclid's Algorithm to determine the:

i) hcf(429, 2919) ii) hcf(3920,7928) iii) hcf(2913, 8274)

Prime Numbers

A prime number is a number that has two factors.

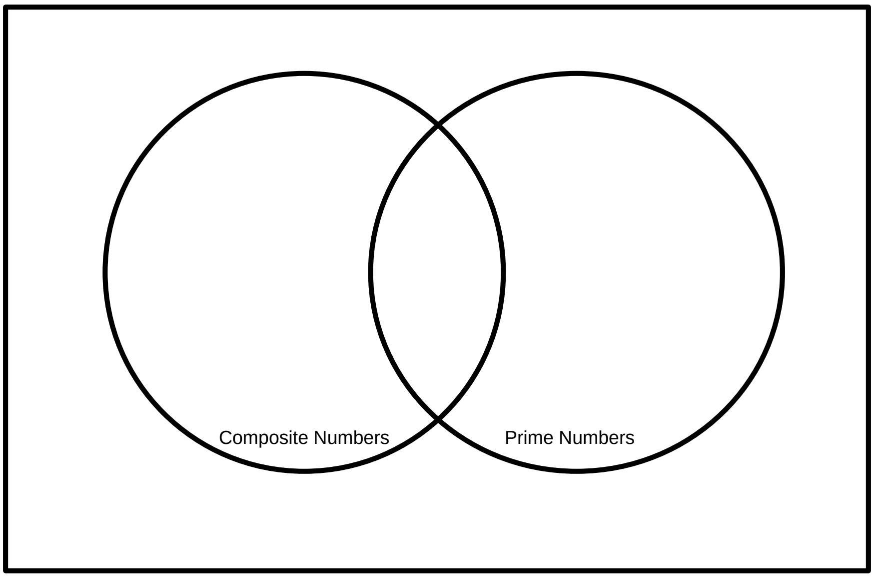
16. a) Is one a prime number?

b) What is special about the number two in relation to prime numbers?

With the exception of 1, a number that is not prime is called a ***composite number***.

17. Copy and complete the following Venn Diagram. Place the following numbers in the correct parts of the diagram.

i) 7 ii) 38 iii) 1 iv) 17 v) 928 vi) 19329 v) 2819



As we have seen, we can split numbers into factors. If a factor happens to be a prime number, it is called a ***prime factor***. All composite numbers can be written as a product of their prime factors. This is called the ***Fundemental Theorem of Arithmetic.***

eg

$243 \rightarrow 3 \times 81 \rightarrow 3 \times 3 \times 27 \rightarrow 3 \times 3 \times 3 \times 9 \rightarrow 3 \times 3 \times 3 \times 3 \times 3 \rightarrow 3^5$

$2000 \rightarrow 2 \times 1000 \rightarrow 2 \times 2 \times 500 \rightarrow 2 \times 2 \times 2 \times 250 \rightarrow 2 \times 2 \times 2 \times 5 \times 50 \rightarrow 2 \times 2 \times 2 \times 5 \times 5 \times 10 \rightarrow 2 \times 2 \times 2 \times 5 \times 5 \times 5 \times 2 \rightarrow 2^4 \times 5^3$

18. Express the following numbers as prime factors.

i) 5932 ii) 9201 iii) 8225 iv) 2819

Eratosthenes' Sieve

If you want to work out all the prime numbers up to and including a particular number, for example, 100, then you only need to divide the numbers by the prime numbers ≤ $\sqrt{100}$ = 10. So this set of prime numbers comprises of {2, 3, 5, 7}.

Eratosthenes devised a way of listing all the prime numbers up to a particular total. He wrote all the numbers between 2 and the total. He then circled the ② and then crossed out all the factors of two up to his total. He then repeated the process for 3, 5 and 7 and ended up with all the prime numbers, which were uncrossed, between 2 and 100. If you want to extend the list further, you will have to check for more numbers but never need to go over the square root of the total number.

19. Find the primes between 2 and 250.

Goldbach Conjecture

Goldbach suggested that every even number greater than 6 can be written as the sum of two primes and every odd number, the sum of three primes.

6=3+3

7=2+2+3

8=5+3

9=2+2+5

10=3+7

11=2+2+7

20. Investigate the Goldbach Conjecture by checking each number between 15 and 45.

The Goldbach Conjecture has never been proven to be true although Vinogradov did prove, in 1937, that every sufficiently large odd integer is the sum of three primes. By sufficiently large, he meant $3^{14348907}$, which is 3×3×3×…14 million, 348 thousand, 900 lots of times 3… ×3×3×3×3. This is a lot!

The number of prime numbers up to a particular number is given a function: $\pi(x)$ where x is the number. So for instance, the number of prime numbers up to 10 would be written as:

$$\pi(10) = 4$$

as there are 4 prime numbers between 1 and 10 (2,3,5,7)

21. Work out the following:

$$i)\ \pi(40) - \pi(20) =$$

$$ii)\ \pi(1000) - \pi(980) =$$

22. Work out how you would investigate whether prime numbers become rarer or more common as you go along the number line. Carry out this investigation.