Divisibility by 9

We show that a positive integer is divisible by 9 if, and only if, the sum of its digits when written in standard decimal notation is also divisible by 9.

The key idea is to think about the meaning of the standard decimal notation.

It helps to begin with a numerical example. For example, the notation "4833" represents the number $4000 + 800 + 30 + 3$, that is, $4 \times 1000 + 8 \times 100 + 3 \times 10 + 3$.

It follows that
$$\begin{aligned} 4833 &= 4 \times (999 + 1) + 8 \times (99 + 1) + 3 \times (9 + 1) + 3 \\ &= (4 \times 999 + 8 \times 99 + 3 \times 9) + (4 + 8 + 3 + 3) . \qquad (1) \end{aligned}$$

Since each of the products 4×999, 8×99 and 3×9 is divisible by 9, the sum $4 \times 999 + 8 \times 99 + 3 \times 9$ is also divisible by 9. Hence, from (1), we see that

4833 is divisible by 9 if, and only if, $4 + 8 + 3 + 3$, is divisible by 9.

It is easy to generalize this argument. We first note that for each positive integer *k*, the number $10^k - 1$ is divisible by 9.

Now let *n* be a positive integer which is written in standard decimal notation as
$$a_k a_{k-1} ... a_2 a_1 a_0$$
where each of a_0, a_1, … $a_{k-1} a_k$ is a digit in the range from 0 to 9.

Then
$$\begin{aligned} n &= a_k \times 10^k + a_{k-1} \times 10^{k-1} + ... + a_2 \times 100 + a_1 \times 10 + a_0 \\ &= a_k \times (10^k - 1 + 1) + a_{k-1} \times (10^{k-2} - 1 + 1) + ... + a_2 \times (99 + 1) + a_1 \times (9 + 1) + a_0 \\ &= (a_k \times (10^k - 1) + a_{k-1} \times (10^{k-1} - 1) + ... + a_2 \times 99 + a_1 \times 9) + (a_k + a_{k-1} + ... + a_2 + a_1 + a_0) . \ (2) \end{aligned}$$

Since each of the products $a_k \times (10^k - 1)$, $a_{k-1} \times (10^{k-1} - 1)$, …, $a_2 \times 99$ and $a_1 \times 9$ is divisible by 9, the sum $(a_k \times (10^k - 1) + a_{k-1} \times (10^{k-1} - 1) + ... + a_2 \times 99 + a_1 \times 9)$ is also divisible by 9. Therefore, by (2),

n is divisible by 9 if, and only if, $a_k + a_{k-1} + ... + a_2 + a_1 + a_0$ is divisible by 9.